

Chapter 5 Analyzing Change: Applications of Derivatives



5.2 Relative and Absolute Extreme Points

Your calculator can be very helpful for checking your analytic work when you find optimal points and points of inflection. When you are not required to show work using derivative formulas or when an approximation to the exact answer is all that is required, it is a simple process to use your calculator to find optimal points and inflection points.

5.2.1 FINDING X-INTERCEPTS OF SLOPE GRAPHS Where the graph of a function has a relative maximum or minimum, the slope graph has a horizontal tangent. Where the tangent line is horizontal, the derivative of the function is zero. Thus, finding where the slope graph *crosses* the input axis is the same as finding the input of a relative extreme point.

Consider, for example, the model for Acme Cable Company's revenue for the 26 weeks after it began a sales campaign, where x is the number of weeks since Acme began sales:

$$R(x) = -3x^4 + 160x^3 - 3000x^2 + 24,000x \text{ dollars}$$

In Example 2 of Section 5.2 of *Calculus Concepts*, we are asked to find when Acme's revenue was greatest during the 26-week interval.

<p>Enter R in the $y1$ location of the $Y=$ list. Enter either the TI-89's derivative or your algebraic derivative in the $y2$ location. Turn off $y1$.</p> <p>(If you use your derivative, be sure to check that your derivative and the TI-89 derivative are the same.)</p>	
<p>The statement of the problem indicates that x should be graphed between 1 and 26. Set this horizontal view, and draw the slope graph in $y2$ with $\boxed{F2}$ [Zoom] $\boxed{\alpha}$ $\boxed{=}$ (A) [ZoomFit]. Reset the window to a better view for this illustration: $y_{min} = -1500$, $y_{max} = 3000$. Redraw the slope graph with \blacklozenge $\boxed{F3}$ (GRAPH).</p>	
<p>With the graph on the screen, find the intercepts of the slope graph with $\boxed{F5}$ [Math] 2 [Zero]. Press and hold \blacktriangleright to move the cursor near to, but still to the <i>left</i> of, the rightmost x-intercept.</p>	
<p>Press \boxed{ENTER} to mark the location of the lower bound. Use \blacktriangleright to move the cursor near to, but to the <i>right</i> of, the rightmost x-intercept. Press \boxed{ENTER} to mark the location of the upper bound.</p>	
<p>Note that the calculator marks the left and right bounds you use with small triangles. The x-intercept must lie between the two small triangle marks. If you incorrectly mark the interval, you may not get an answer. The coordinates of the x-intercept (zero) of the slope graph are displayed: $R'(x) = 0$ at $x = 20$.</p>	

We now need to determine if the slope graph crosses the x -axis, only touches the x -axis, or does neither of these at the other location that may be an intercept.

<p>Zoom in with ZoomIn (see Section 1.4.2) or ZoomBox (see Section 3.2.1) as many times as necessary to magnify the portion of the graph around $x = 20$ in order to examine it more closely.</p> <p>We choose to use ZoomBox, but both work equally well.</p>	
<p>After magnifying the graph several times, we see that the graph just touches and does not cross the x-axis near $x = 10$. (Press $\boxed{F3}$ [Trace] and trace as near as possible to $x = 10$.) We see that $x = 10$ does not yield an extreme point on the graph of R.</p>	
<p>We are asked to find the absolute maximum, and we know that it occurs at one of the endpoints of the interval or at a zero of the slope graph. So, return to the home screen and find the outputs of R at the endpoints of the interval ($x = 0$ and $x = 26$) as well as the output at the “crossing” root of the slope graph ($x = 20$).</p>	

We see that Acme’s revenue was greatest at 20 weeks after they began the sales campaign.

5.2.2 FINDING ZEROS OF SLOPE FUNCTIONS USING THE SOLVER You may find it more convenient to use the TI-89’s solve instruction rather than find the x -intercept(s) of the slope graph as we did in the last section. We illustrate using Acme Cable’s revenue function.

<p>Enter R in the $y1$ location of the $Y=$ list. Enter either the TI-89’s derivative or your derivative in the $y2$ location. Turn off $y2$.</p> <p>Recall that the slope graph crosses the input axis at either a max or a min of the function graph. Viewing a graph of the function is an easy way to tell which extrema occur at which locations.</p>	
<p>We are told that x is between 1 and 26, so set this horizontal view. Draw the graph of $y1$ with $\boxed{F2}$ [Zoom] α $\boxed{=}$ (A) [ZoomFit].</p>	
<p>Reset y_{max} to a larger value, say 95,000, to better see the high point on the graph. Graph R and press $\boxed{F3}$ [Trace]. Hold down \blacktriangleright until you have an estimate of the input location of the high point. The maximum seems to occur when x is near 20.</p>	

Remember that we want to solve the equation $R'(x) = 0$ to find where the tangent is horizontal.

<p>On the home screen, type the solve instruction with $\boxed{F2}$ [Algebra] 1 [solve()], type “$y2(x) = 0, x$”, and press \boxed{ENTER}. The solutions $x = 10$ and $x = 20$ are found.</p>	
--	--

- The TI-89 is correct because the slope is 0 at both of these inputs. However, as we saw from the graph of the function R , we cannot rely only on the results of the solve instruction. The function graph should also be examined. In this case, we see that only 20 is the location of the maximum. Acme's revenue was greatest at 20 weeks after they began the sales campaign.

5.2.3 USING THE CALCULATOR TO FIND OPTIMAL POINTS Once you draw a graph of a function that clearly shows the optimal points, your calculator can find the location of those high points and low points without using calculus. However, we recommend not relying only on this method because your instructor may ask you to show your work using derivatives. If so, this method would probably earn you no credit! This method does give a good check of your answer, and we illustrate it using Acme Cable's revenue function R from Section 5.2.1.

<p>Enter R in the $y1$ location of the $Y=$ list. We are told that x is between 1 and 26, so set this horizontal view. Draw the graph of $y1$ with $\boxed{F2}$ [Zoom] $\boxed{\alpha}$ $\boxed{=}$ (A) [ZoomFit]. Reset y_{max} to a larger value, say 95,000, to leave more room above the high point on the graph.</p>	
<p>With the graph of R on the screen, press $\boxed{F5}$ [Math] 4 [Maximum]. Use $\boxed{\blacktriangleright}$ or $\boxed{\blacktriangleleft}$ to move the cursor near, but still to the <i>left</i> of, the high point on the curve. Press \boxed{ENTER} to mark the lower bound of the interval.</p>	
<p>Use $\boxed{\blacktriangleright}$ to move the cursor to the <i>right</i> of the high point. Press \boxed{ENTER} to mark the upper bound of the interval and find the relative maximum point.</p>	

CAUTION: As when using the Math menu to find zeros, the bounds you mark are indicated on the graph with small triangles. When you enter your estimate for the upper bound of the interval, the TI-89 returns the highest point that is within the bounded interval that you have marked. If the relative maximum does not lie in this interval, the TI-89 returns the highest point in the interval you marked (usually an endpoint). Always look at the function graph!

- You may not see the exact answer for either one or both of the coordinates at the maximum point due to rounding errors in the numerical routine used by the TI-89. Always round your answer to make sense in the context.
- The method shown in this section also applies to finding the relative minimum value(s) of a function. The only difference is that to find the relative minimum instead of the relative maximum, initially press $\boxed{F5}$ [Math] 3 [Minimum] from the graph screen.

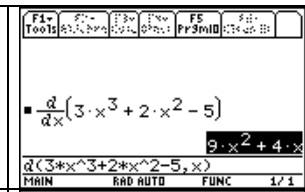
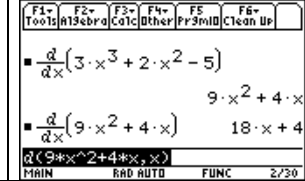
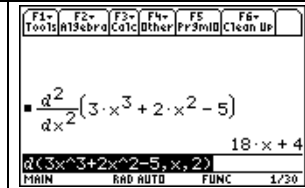


5.4 Inflection Points

As was the case with optimal points, your calculator can be very helpful in checking algebraic work when you find points of inflection. You can also use the methods illustrated in Section 5.2.3 of this *Guide* to find the location of any maximum or minimum points on the graph of the first derivative to find the location of any inflection points for the function.

5.4.1 FINDING X-INTERCEPTS OF A SECOND-DERIVATIVE GRAPH We first look at using the algebraic method of finding inflection points – finding where the graph of the second derivative of a function *crosses* the input axis.

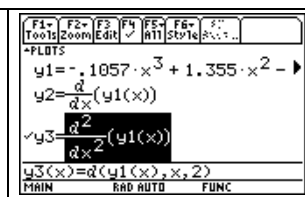
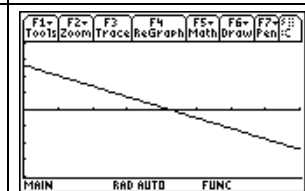
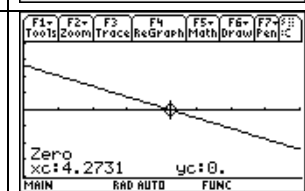
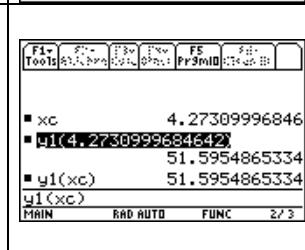
Before illustrating with an example, we see how to find second derivatives on the TI-89.

<p>Find the first derivative of a function, say $f(x) = 3x^3 + 2x^2 - 5$. Highlight the first derivative and press [F1] [Tools] 5 [Copy].</p>	
<p>Use [▼] to return to the entry line and delete the function with [←]. Use [F1] [Tools] 6 [Paste] to paste the first derivative in that location and press [ENTER].</p>	
<p>Another method allows you to find the first derivative directly. Enter a third value into the derivative instruction (following the input variable) where the value entered is a number, called the <i>order</i>, corresponding to the derivative you want. For a second derivative, we enter 2.</p>	

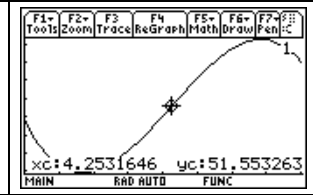
To illustrate, we consider a model for the percentage of students graduating from high school in South Carolina from 1982 through 1990 who entered post-secondary institutions:

$$f(x) = -0.1057x^3 + 1.355x^2 - 3.672x + 50.792 \text{ percent}$$

where x is the number of years after 1982.

<p>Enter f in the $y1$ location of the $Y=$ list, the first derivative of f in $y2$, and the second derivative of f in $y3$. (Note that you can find these on the home screen, copy them, and paste the algebraic formulas into the $Y=$ list or you can use the derivative symbol notation that is shown to the right.) Turn off $y1$ and $y2$.</p>	
<p>We are given the input interval 1982 through 1990, so $0 \leq x \leq 8$. Either use ZoomFit or choose some appropriate vertical view. We use $-4 \leq y \leq 4$. Because we are looking for the x-intercept(s) of the second derivative graph, any view that shows the line crossing the horizontal axis is okay to use.</p>	
<p>Use the methods indicated in Section 5.2.1 to find where the second derivative graph crosses the x-axis. (Note that when you are asked to give the inflection <i>point</i> of f, you should answer with both the input and an output of the original function.)</p>	
<p>Return to the home screen and enter [X] [alpha] [)] (c). The x-value you just found as the x-intercept remains stored in the xc location until you change it by tracing, using the solve instruction, and so forth. Find the function output by substituting this x-value into $y1$. Note that the highlighted instruction to the right was entered with [Y] 1 [2nd] [−] (ANS).</p>	

At some point, be sure to examine a graph of the function and verify that an inflection point does occur at the point you have found. To do this, turn off y3, turn on y1, and use ZoomFit to draw the graph. Trace near where $x \approx 4.27$ and $y \approx 51.6$. The graph of R confirms that an inflection point occurs at this point.

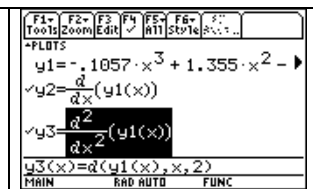


EXPLORE: What would the line tangent to the graph of f look like at the inflection point? Use the graph of $y1$ and the first method explained on page C-50 of this *Guide* (with $x = 4.27$) to see if you are correct.

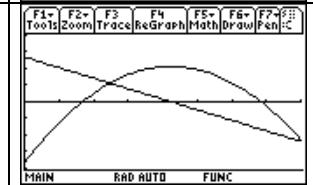
The TI-89 usually draws an accurate graph of the first derivative of a function when you use the derivative notation to specify the derivative rather than use an algebraic formula. Your calculator also draws the graph of and computes the second derivative of a function when you specify the order of the derivative in the notation. If you are in doubt about your calculation of the first and second derivative formulas, check them using the TI-89's symbolic capabilities or use the calculator's derivative notation rather than your algebraic formula to avoid making a mistake in the location of the inflection point.

We continue the illustration using the South Carolina high school graduation rate function.

You should still have f in the $y1$ location of the $Y=$ list, the first derivative of f in $y2$, and the second derivative of f in $y3$. Turn off $y1$ and turn on $y2$ and $y3$. Check the window; you should still have $xmin = 0$ and $xmax = 8$.



Draw the graph of the first derivative ($y2$) and the second derivative ($y3$) of f using an appropriate window. You can use ZoomFit to set the vertical view or experiment until you find a suitable view. The graph to the right uses $-4 \leq y \leq 4$.



Find the x -intercept of the second derivative graph as indicated in this section or find the input of the high point on the first derivative graph (see below) to locate the input of the inflection point. (When more than one graph is on the screen, use \blacktriangledown to jump between graphs.)

5.4.2 USING THE CALCULATOR TO FIND INFLECTION POINTS Remember that an inflection point on the graph of a function is a point of greatest or least slope. This means that you can find the input location of an inflection point by finding where the first derivative of the function has a maximum or minimum slope. We illustrate this method using the logistic function for polio cases that is in Example 2 of Section 5.4 in *Calculus Concepts*:

$$C(t) = \frac{42,183.911}{1 + 21,484.253e^{-1.248911t}}$$

where $t = 1$ in January, $t = 2$ in February, and so forth.

Enter $C(x) = 42183.911/(1 + 21484.253e^{(-1.248911x)})$ in the $y1$ location of the $Y=$ list and the first derivative of f in $y2$. (You can use your algebraic formula for the first derivative or the calculator's derivative.) Turn off $y1$.



<p>The problem context says that the input interval is from 0 (the beginning of 1949) to 12 (the end of 1949), so set these values for x in the window. Set the vertical view and draw the graph of C' with $\boxed{F2}$ [Zoom] $\boxed{\alpha}$ $\boxed{=}$ (A) [ZoomFit].</p>	
<p>Use the method discussed in Section 5.2.3, <i>i.e.</i>, use $\boxed{F5}$ [Math] 4 [Maximum], to find the input location of the maximum point on the slope graph. (<i>Note:</i> This graph takes a while to draw because the TI-89 must calculate each derivative before plotting the point. Tracing this graph is also slower than if you had entered an algebraic formula for C'.)</p>	
<p>The x-value of the maximum of the slope graph is the x-value of the inflection point of the function. To find the rate of change of polio cases at this time, substitute this value of x in $y2$. To find the number of cases at this time, substitute x in $y1$.</p>	
<p>You could have found the input of the inflection point on the polio cases graph by finding the x-intercept of the second derivative graph. The function whose graph is shown to the right is $y3 = d(y1(x), x, 2)$, and the graph was drawn using ZoomFit.</p>	
<p>If you prefer, you could have found the input of the inflection point by solving the equation $C'' = d(y1(x), x, 2) = 0$ using the solve instruction.</p>	

CAUTION: Do not forget to round your answers appropriately (this function should be interpreted discretely) and to give units of measure with each answer.

Our final method is also the simplest – using the TI-89’s built-in inflection point finder! Be certain if you use this method that the function does have an inflection point at the location indicated by the calculator. We illustrate with the polio cases function that is entered in $y1$: $C(x) = 42183.911/(1 + 21484.253e^{(-1.248911x)})$.

<p>Turn $y1$ on and all other functions off. Draw a graph of C for x between 0 and 12 with ZoomFit. Press $\boxed{F5}$ [Math] 8 [Inflection]. Move, using $\boxed{\blacktriangleright}$ or $\boxed{\blacktriangleleft}$, the cursor to a position to the left of where the function changes concavity.</p>	
<p>\boxed{ENTER} marks the lower bound. Use $\boxed{\blacktriangleright}$ to move the cursor to the right of where the function changes concavity. Press \boxed{ENTER} and the input and output of the inflection point are shown.</p>	