Chapter 3 Describing Change: Rates

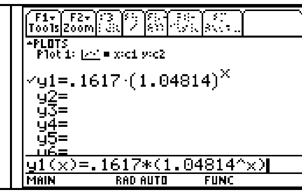
3.1 Change, Percent Change, and Average Rates of Change

As you calculate average and other rates of change, remember that every numerical answer in a context should be accompanied by units telling how the quantity is measured. You should also be able to interpret each numerical answer. It is only through their interpretations that the results of your calculations will be useful in real-world situations.

3.1.1 FINDING AVERAGE RATE OF CHANGE Finding an average rate of change[1] using a function is just a matter of evaluating the equation at two different values of the input variable and dividing by the difference of those input values.

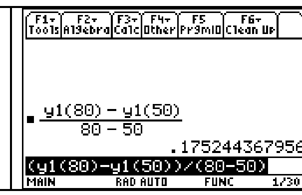
We illustrate this concept using the function describing the population density of Nevada from 1950 through 1997 that is given in Example 3 of Section 3.1 of *Calculus Concepts*.

The population density referred to above is given by the function $p(t) = 0.1617(1.04814^t)$ people per square mile t years after 1900. Press ◆ F1 (Y=), clear any functions, turn Plot 1 off, and enter this function in y1 using x as the input variable.



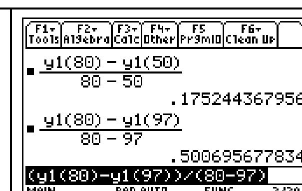
To find the average rate of change of the population density from 1950 through 1980, first realize that 1950 corresponds to $x = 50$ and 1980 corresponds to $x = 80$. Then, recall that the average rate of change of p between 1950 and 1980 is given by the quotient $\frac{p(80)-p(50)}{80-50}$.

Return to the home screen and type the quotient $\frac{y1(80)-y1(50)}{80-50}$. Remember to enclose both the numerator and denominator of the fraction in parentheses. Finding this quotient in a single step avoids having to round intermediate calculation results.

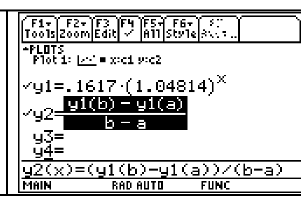


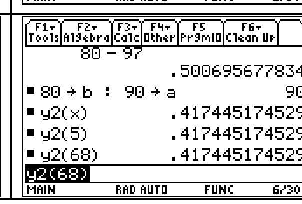
Recall that rate of change units are output units per input units. On average, Nevada's population density increased about 0.18 people per square mile per year between 1950 and 1980.

To find the average rate of change between 1980 and 1997, press ▶ and change 50 to 97 in two places. Press ENTER. If you have several average rates of change to calculate, you can put the average rate of change formula in the graphing list. Of course, you need to have a function in y1.



Enter y2 = (y1(b) – y1(a))/(b – a). On the home screen, store the inputs of the two points in a and b. Type y2(x) (or insert any number for x because y2 is a constant – it does not contain x).



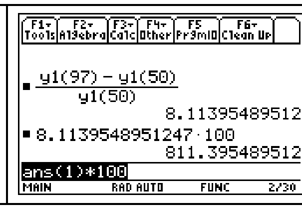


3.1.2 CALCULATING PERCENTAGE CHANGE You can find percentage changes using data either by the formula or by using program DIFF. To find a percentage change from a function

[1] The TI-89 CATALOG contains an instruction called *avgRC*. While this instruction could be adapted to give what we call the *average rate of change*, it is easer to calculate average rate of change as illustrated in this *Guide*.

instead of data, you should use the percentage change formula. We again use the population density of Nevada function in Example 3 of Section 3.1 of *Calculus Concepts* to illustrate.

Have $p(x) = 0.1617(1.04814^x)$ in y1. The percentage change in the population density between 1950 and 1997 is given by the formula $\frac{p(97)-p(50)}{p(50)}\cdot 100\% = \frac{\text{y1}(97)-\text{y1}(50)}{\text{y1}(50)}\cdot 100\%$. Recall that if you type in the entire quotient, you avoid rounding errors.



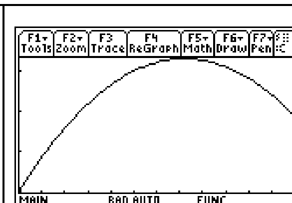
The population density of Nevada increased by about 811.4% between 1950 and 1997.

3.2 Instantaneous Rates of Change

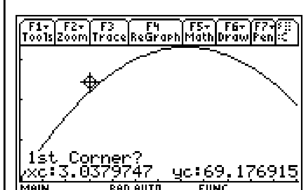
We first examine the principle of local linearity, which says that if you are close enough, the tangent line and the curve are indistinguishable. We also explore the method the calculator uses to draw a tangent line at a point on a curve.

3.2.1 MAGNIFYING A PORTION OF A GRAPH The zoom menu of your calculator allows you to magnify any portion of a graph. Consider the graph shown in Figure 3.7 in Section 3.2 of *Calculus Concepts*. The temperature model is $T(x) = -0.804x^2 + 11.644x + 38.114$ degrees Fahrenheit where x is the number of hours after 6 a.m.

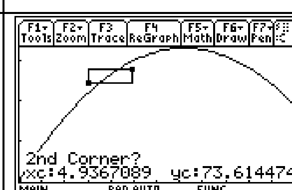
Enter the function T in y1, set the window for x between 6 a.m. ($x = 0$) and 6 p.m. ($x = 12$), and draw the graph using ◆ F1 (Y=) F2 [Zoom] alpha = (A) [ZoomFit]. We now want to zoom and "box" in several points on this graph to see a magnified view.



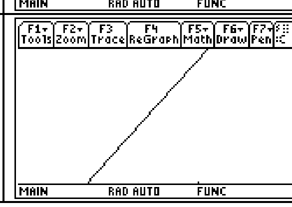
The first point we consider on the graph is point A with $x = 3$. Press F2 [Zoom] 1 [ZoomBox] and use the arrow keys (◄, ▲, etc.) to move the cursor to the left of the curve close to where $x = 3$. (You may not have the same coordinates as those shown on the right.) Press ENTER to fix the lower left corner of the box.



Use the arrow keys to move the cursor to the opposite corner of your "zoom" box. Point A should be close to the center of your box. Press ENTER to magnify the portion of the graph that is inside the box.



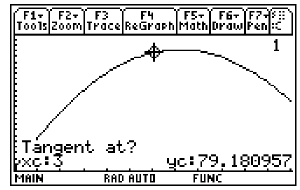
Look at the dimensions of the view you now see with ◆ F2 (WINDOW). Repeat the above process if necessary. The closer you zoom in on the graph, the more it looks like a line.



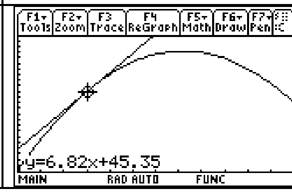
Reset xmin to 0, xmax to 12, and redraw the graph of the temperature function with ZoomFit. Repeat the zoom and box steps for point B with $x = 7.24$ and point C with $x = 9$. Note for each point how the graph of the parabola looks more like a line the more you magnify the view. These are illustrations of *local linearity*.

3.2.2 DRAWING A TANGENT LINE The TI-89 has several different instructions that allow you to draw a line tangent to the graph of a function at a specified point. To illustrate the process, we draw a line tangent to the graph of T, the temperature function given in the previous section of this *Guide*.

Enter $T(x) = -0.804x^2 + 11.644x + 38.114$ in y1. Next, press [◆] [F2] (WINDOW) and set xmin = 0, xmax = 12, ymin = 38, and ymax = 85. Press [◆] [F3] (GRAPH). Press [F5] [Math] [alpha] [=] (A) [Tangent].



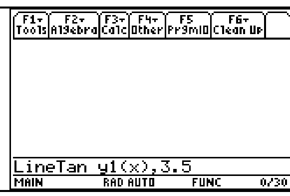
Press 3 [ENTER] at the "Tangent at?" prompt, and the tangent line is drawn at $x = 3$.

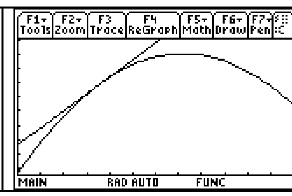


Note that the equation of the tangent line is displayed at the bottom of the screen. Compare this line with the zoomed-in view of the curve at point A (in the previous section). They are actually the same line if you are at point A!

The TI-89 instruction LineTan also allows you to draw a line tangent to a curve at a given point. Access this instruction from the CATALOG. Whenever you use this instruction, the calculator assumes that the graph of a function is already drawn using x as the input variable. Press [◆] [F3] (GRAPH). If the line tangent to the temperature curve at $x = 3$ is still drawn, press [F4] [ReGraph]. Return to the home screen with [HOME].

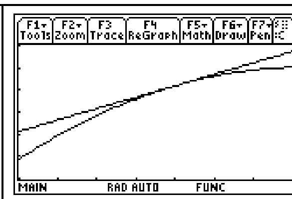
Press [CATALOG] 4 and scroll down to LineTan. Press [ENTER]. Next type y1(x) and the input of the point of tangency, separated by a comma.





Your calculator has yet another instruction called DrawSlp that can be used to draw a line tangent to a curve if you know the slope of the line at the point of tangency. Suppose you are told that the slope of the line tangent to the temperature curve at $x = 4$ is 5.212. Let us use this information to draw the tangent line at the point on the curve where $x = 4$.

Press [◆] [F3] (GRAPH). If the tangent line to the temperature curve at $x = 3$ is still drawn, press [F4] [ReGraph]. Press [2nd] [F1] [F6: Draw] 6 [DrawSlp] 4 [,] [Y] 1 [(] 4 [)] [,] 5.212 [ENTER] .

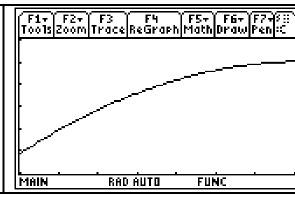


NOTE: The DrawSlp instruction returns you to the home screen but the tangent line automatically draws when you enter the complete instruction.

3.2.3 VISUALIZING THE LIMITING PROCESS This section of the *Guide* is optional, but it might help you understand what it means when we say that the tangent line is the limiting position of secant lines. Program SECTAN can be used to view secant lines between a point $(a, f(a))$ and close points on a curve $y = f(x)$. The program draws the tangent line at the point $(a, f(a))$. Before using program SECTAN, a function (using x as the input variable) must be entered in the y1 location of the Y= list, and you must draw a graph of the function. We illustrate with the graph of the function T that was used in the previous two sections, but you can use this program with any graph. (Program SECTAN is in the *TI-89 Program Appendix*.)

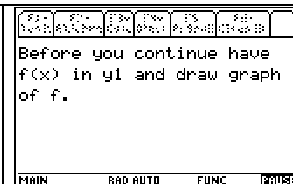
Caution: In order to properly view the secant lines and the tangent line, it is essential that you first draw a graph of the function clearly showing the function, the point of tangency (which should be near the center of the graph), and enough space so that the close points on either side can be seen. See Sections 1.1.2 and 1.1.3 of this *Guide* for some hints on finding a proper viewing window.

Have $T(x) = {}^-0.804x^2 + 11.644x + 38.114$ in y1. Next, press ◆ F2 (WINDOW) and set xmin = 0, xmax = 7, ymin = 30, and ymax = 90. Press ◆ F3 (GRAPH) and then use HOME to return to the home screen.

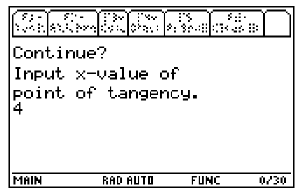


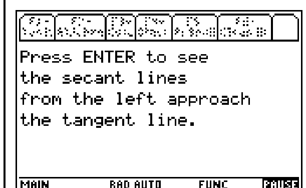
Press 2nd – (VAR-LINK) 3 (S) and scroll to "sectan" (or type the name on the home screen).

Press ENTER to start the program and see the message that appears to the right. Press ENTER. If you forgot to enter the function in y1 or to draw its graph, press 2 [no] at the Choice? prompt that appears. Otherwise, press 1 [yes].

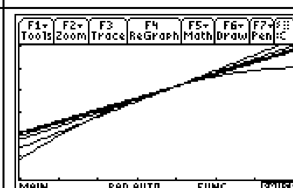


At the next prompt, type the x-value at the point of tangency. (For this illustration, choose $x = 4$.) When ENTER is pressed you see the message shown on the far right.

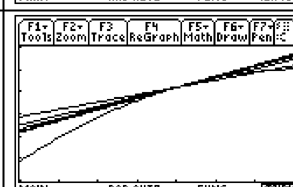




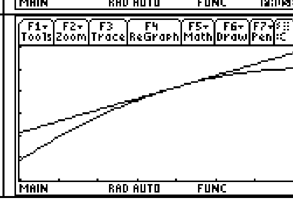
Press ENTER. Five secant lines will draw. (Remember that the indicator at the lower right-hand corner of the screen says BUSY when the program is computing and says PAUSE when it is finished and ready for you to press ENTER.)



When you finish looking at the graph, press ENTER to continue. A message appears and says that you will see next secant lines drawn between the point of tangency and close points to the *right*. (Again, five secant lines will draw.) Press ENTER.



Press ENTER to continue the program. You are next told to press ENTER to see a graph of the line that is tangent to the curve at $x = 4$.



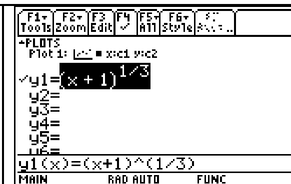
As you watch the graphs, you should notice that the secant lines are becoming closer and closer to the tangent line as the close point moves closer and closer to the point of tangency.

3.2.4 TANGENT LINES AND INSTANTANEOUS RATES OF CHANGE

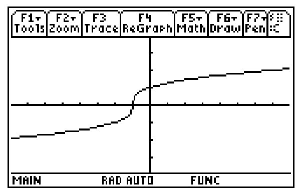
The TI-89 usually gives results that are the same as the mathematical results you expect. However, as with all technologies, you should now the fundamentals of what results to expect just in case the calculator is not correct for some particular result. In this section, we investigate what the calculator does if you ask it to draw a tangent line where the line cannot be drawn. All plots should be turned off. Consider these special cases:

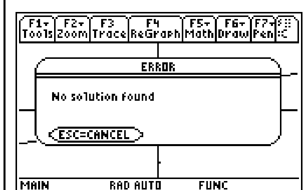
1. What happens if the tangent line is vertical? We consider the function $f(x) = (x + 1)^{1/3}$ which has a vertical tangent at $x = -1$.
2. How does the calculator respond when the instantaneous rate of change at a point does not exist? We illustrate with $g(x) = |x| - 1$, a function that has a sharp point at $(0, -1)$.
3. Does the calculator draw the tangent line at the break point(s) of a piecewise continuous function? We consider two situations:
   a. $h(x)$, a piecewise continuous function that is continuous at all points, and
   b. $m(x)$, a piecewise continuous function that is not continuous at $x = 1$.

1. Enter the function $f(x) = (x + 1)^{1/3}$ in the y1 location of the Y= list. Remember that anytime there is more than one symbol in an exponent and you are not sure of the TI-89's order of operations, enclose the power in parentheses.



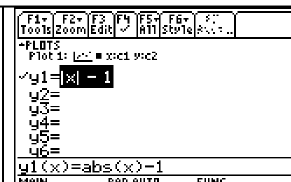
Draw the graph of f with [F2] [Zoom] 4 [ZoomDec]. Press [F5] [Math] [alpha] [=] (A) [Tangent] and [(–)] 1 [ENTER] at the "Tangent at?" prompt.



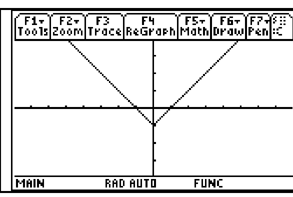


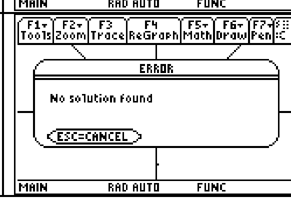
CAUTION: A vertical tangent line at $x = -1$ does not draw (as it should). For this function, the slope of the tangent line does not exist, but the tangent line can be drawn. Note that you can manually draw the tangent using the vertical line instruction. From the home screen access LineVert from the CATALOG and enter the instruction LineVert −1.

2. Clear y1 and enter the function $g(x) = |x| - 1$. The absolute value symbol is typed with [2nd] 5 (MATH) 1 [Number] 2 [abs(]. Draw the graph of the function with [F2] [Zoom] 4 [ZoomDec].



Press [F5] [Math] [alpha] [=] (A) [Tangent] and 0 [ENTER] at the "Tangent at?" prompt.

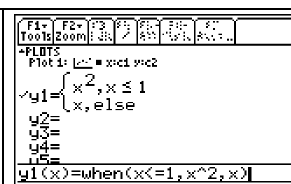




This is correct! There is a sharp point on the graph of g at $(0, -1)$, and the limiting positions of secant lines from the left and the right at that point are different. A tangent line cannot be drawn on the graph of g at $(0, -1)$ according to our definition of instantaneous rate of change.

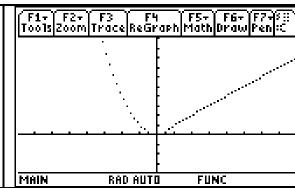
3a. Clear any functions from in the Y= list. Enter, as indicated, the function

$$h(x) = \begin{cases} x^2 & \text{when } x \le 1 \\ x & \text{when } x > 1 \end{cases}$$

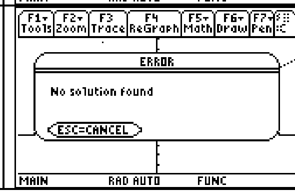


Recall that *when* is found in the CATALOG and that the < symbol is accessed with [2nd] 0 (<).

Highlight y1 and press [2nd] [F1] [F6: Style] 2 [Dot]. Draw the graph of h with [F2] [Zoom] 4 [ZoomDec]. Reset ymax to 10 for a better view and redraw the graph with [◆] [F3] (GRAPH).



Press [F5] [Math] [alpha] [=] (A) [Tangent] and 1 [ENTER] at the "Tangent at?" prompt.



This is correct! Even though h is continuous for all values of x, there is no tangent line to the graph of h at the break point $x = 1$. Secant lines drawn using close points on the right and on the left of $x = 1$ do not approach the same slope, so the instantaneous rate of change of h does not exist at $x = 1$.

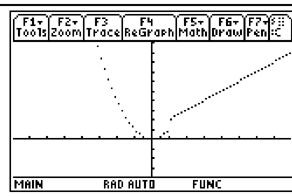
NOTE: There is <u>no tangent line at the break point</u> of a piecewise continuous function (even if that function is continuous at the break point) unless secant lines drawn through close points to the left and right of that point approach the same value.

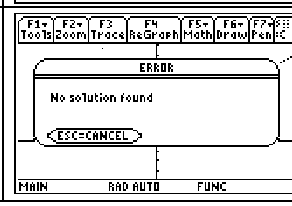
3b. Edit y1 so that the second part of it is $x + 1$ instead of x:

$$m(x) = \begin{cases} x^2 & \text{when } x \le 1 \\ x+1 & \text{when } x > 1 \end{cases}$$

The TI-89 should still be in DOT mode from the previous graph.

Draw the graph of the function with [◆] [F3] (GRAPH). Press [F5] [Math] [alpha] [=] (A) [Tangent] and 1 [ENTER] at the "Tangent at?" prompt.





This is correct! Because m is not continuous at $x = 1$, the instantaneous rate of change does not exist at that point. The tangent line cannot be drawn on the graph of m when $x = 1$.

NOTE: None of the other TI-89 tangent line instructions draw the tangent at the points where the one we used above did not because they all make use the slope of the curve at those points.

3.3 Derivatives

There are no new calculator techniques in this section, but we illustrate a new calculation.

3.3.1 CALCULATING PERCENTAGE RATE OF CHANGE Suppose the growth rate of a population is 50,000 people per year and the current population size is 200,000 people.

What is the percentage rate of change of the population? The answer is 25% per year.

Suppose instead that the current population size is 2 million. What is the percentage rate of change? The answer is 2.5% per year, which is a much smaller percentage rate of change.