Chapter 11 Dynamics of Change: Differential Equations and Proportionality

The TI-89 offers many opportunities when studying differential equations. It draws slope field graphs and has as built-in commands many of the techniques that must be programmed into other calculators. However, in this *Guide*, we look only at those tools that are necessary for the study of differential equations as presented in Chapter 11 of *Calculus Concepts*. We encourage you to further explore the TI-89 capabilities by reading and working through pages 164-185 of your *TI-89 Guidebook*.

11.3 Numerically Estimating by Using Differential Equations: Euler's Method

Many of the differential equations we encounter have solutions that can be found by determining an antiderivative of a given rate-of-change function. Thus, many of the techniques that we learned using the TI-89's integration function apply to this chapter. (See Chapters 6 and 7 of this *Guide*.)

11.3.1 **EULER'S METHOD FOR A DIFFERENTIAL EQUATION WITH ONE INPUT VARIABLE** You may encounter a differential equation that cannot be solved by standard methods and you may need to draw an accumulation graph for a differential equation without first finding an antiderivative. In either of these cases, numerically estimating a solution using Euler's method is helpful. This method relies on the use of the derivative of a function to approximate the change in that function. Recall from Section 5.1 of *Calculus Concepts* that the approximate change in f at a point is the rate of change of f at that point times a small change in x. That is,

$$f(x+h) - f(x) \approx f'(x)\cdot h \quad \text{where } h \text{ represents the small change in } x.$$

Now, if we let $b = x + h$ and $x = a$, the above expression becomes

$$f(b) - f(a) \approx f'(a)\cdot(b-a) \quad \text{or} \quad f(b) \approx f(a) + (b-a)\cdot f'(a)$$

The starting values for the coordinates of the point (a, b) will be given to you and are often called the *initial condition*. The next step is to repeatedly apply the formula given above to use the slope of the tangent line at $x = a$ to approximate the change in the function between the inputs a and b. When h, the distance between a and b, is fairly small, Euler's method will often give close numerical estimates of points on the solution to the differential equation containing $f'(x)$.

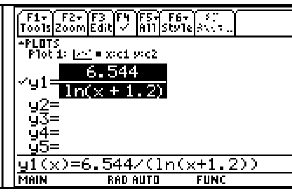
WARNING: Be wary of the fact that there is some error involved in each step of the Euler approximation process that is compounded when each result is used to obtain the next result.

We illustrate Euler's method for a differential equation containing one input variable with the differential equation in Example 1 of Section 11.3. This equation gives the rate of change of the total sales of a computer product t years after the product was introduced:

$$\frac{dS}{dt} = \frac{6.544}{\ln(t+1.2)} \text{ billion dollars per year}$$

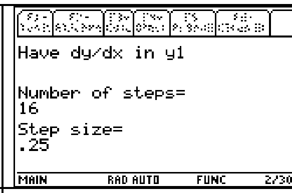
Because Euler's method involves a repetitive process, a program that performs the calculations used to find the approximate change in the function can save you time and eliminate computational errors and some error in rounding.

Before using this program, you must have the differential equation in the y1 location of the Y= list with x as the input variable. On the home screen, type eulr() in the entry line to call the Euler program.

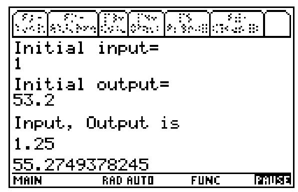


The code for program EULR is listed in the *TI-89 Program Appendix*. Run the program. Each time the program stops for input or for you to view a result, press ENTER to continue.

We choose to use 16 steps. Enter this value. The interval is 4 years, so enter the step size = $\frac{\text{length of interval}}{\text{number of steps}} = \frac{4}{16}$ = 0.25.



The initial condition is given as the point (1, 53.2). Enter these values when prompted for them.

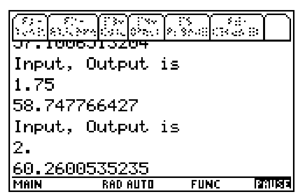


The first application of the formula gives an estimate for the value of the total sales at x = 1.25:

$S(1.25) \approx 55.275$

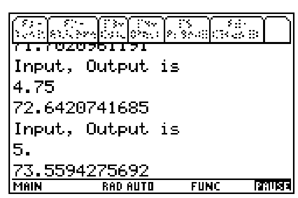
Press ENTER several more times to obtain more estimates for total sales.

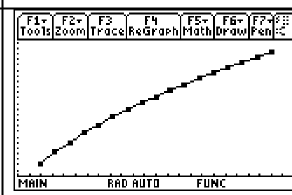
Record the input values and output estimates on paper as the program displays them.



Continue pressing the ENTER key to obtain more estimates of points on the total sales function S.

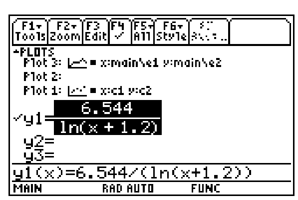
When 16 steps have been completed (that is, after the input reaches 5), the program draws a graph of the points *(input, output estimate)* connected with straight line segments. This graph is an estimate of the graph of the differential equation solution.

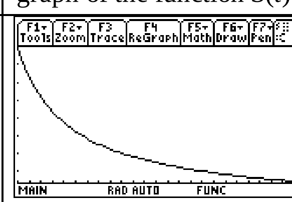




This is an estimate of the graph of the function $S(t)$.

In the Y= list, turn Plot 3 off, turn y1 on, and press F2 [ZOOM] alpha = (A) [ZoomFit] to draw the graph of the differential equation.





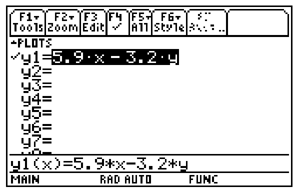
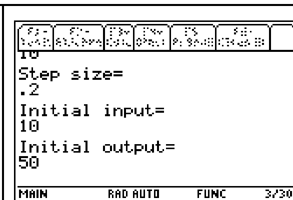
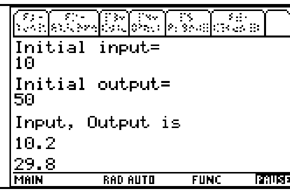
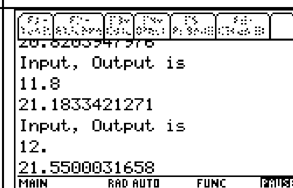
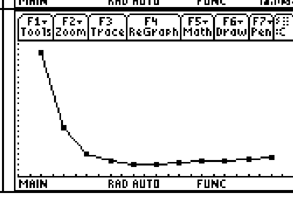
This is the slope graph – the graph of $S'(t)$.

11.3.2 EULER'S METHOD FOR A DIFFERENTIAL EQUATION WITH TWO INPUT VARIABLES Program EULR can be used when the differential equation is a function of x and y with $y = f(x)$. Follow the same process that is illustrated in the previous section of this *Guide*, but enter $\frac{dy}{dx}$ in y1 using the letters x and y as they are written in the given equation.

If the differential equation is written in terms of variables other than x and y, let the derivative symbol be your guide as to which variable corresponds to the input and which corresponds to the output. For instance, if the rate of change of a quantity is given by $\frac{dP}{dn} = 1.346P(1 - n^2)$, compare $\frac{dy}{dx}$ to $\frac{dP}{dn}$, entering in y1 the expression 1.346 y (1 – x ^{2}).

The differential equation may be given in terms of y only. For instance, if $\frac{dy}{dx} = k(30 - y)$ where k is a constant, enter y1 = K(30 – y). Of course, you need to store a value for k or substitute a value for k in the differential equation before using program EULR. It is always better to store the exact value for a constant instead of using a rounded value.

We illustrate using Euler's method with two input variables by using the equation in Example 2 of Section 11.3 in *Calculus Concepts*.

<table>
<tr><td>Enter $\frac{dy}{dx} = 5.9x - 3.2y$ in y1. Run program EULR. We choose to use 10 steps over an interval of length 2; hence, the step size is 0.2. The initial condition is $y(10) = 50$.</td><td>
</td><td>
</td></tr>
<tr><td>The first estimate for a point on the solution is $y = 29.8$ when $x = 10.2$.
Continue until 10 steps are completed; that is, until the input is 12.</td><td>
</td><td>
</td></tr>
<tr><td colspan="2">The estimate of $y(12)$ is 21.55. Press ENTER to draw a graph of the Euler estimates. (Note that we cannot graph the differential equation on the TI-89 using the function graphing mode as we do when there is only one input variable.)</td><td>
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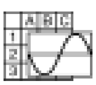
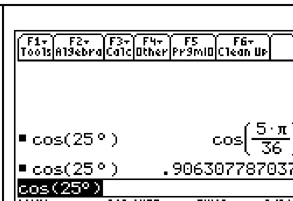
Appendix: Trigonometry Basics



Before you begin working through the ideas in this *Appendix*, be certain that page 1 of the MODE screen in the basic setup has Radian/Degree set to Radian measure.

A.1 FINDING OUTPUTS OF TRIG FUNCTIONS WITH DEGREE INPUTS It is very important that you have the correct mode set when evaluating trigonometric function outputs. Unless you see the degree symbol in a problem, the mode setting should be Radian. We show two different methods of evaluating trig functions with degree inputs, using the illustration as given in the discussion following Figure A9 in the *Trigonometry Appendix*.

<table>
<tr><td>To find cos 25° with MODE set to Radian, press 2nd Z 25 and then tell the TI-89 that the angle is measured in degrees by pressing 2nd 5 (MATH) 2 [Angle] 1 (°) ENTER. The TI-89 expresses the cosine by writing the angle in radians. To find the decimal approximation, press ◆ ENTER.</td><td>
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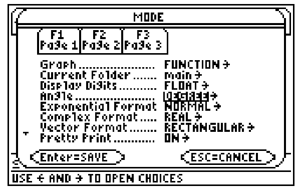
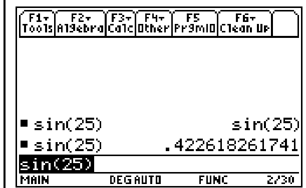
</table>

WARNING: The next method changes the MODE setting to Degree. Do not forget to change it setting back to Radian if you use this second method. If not, you will get an incorrect value the next time you evaluate a trig function for an angle measured in radians or when drawing a graph. We recommend using the first method for this reason.

Now find sin 25^o with another method: change page 1 of the MODE setting to Degree, return to the home screen, type [SIN] 25, and press [ENTER]. Press [◆] [ENTER] for the decimal value.

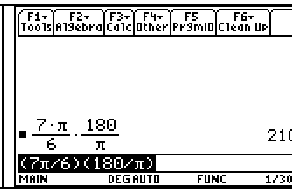




A.2 CONVERTING ANGLES FROM RADIANS TO DEGREES

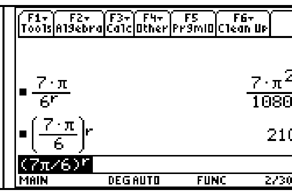
Even though the formula for this conversion is simple, the TI-89 converts angles from radians to degrees quite easily. We illustrate using the angle in part *a* of Example 7 in the *Trigonometry Appendix*.

As shown in Example 7, convert 7π/6 radians to degrees with multiplication by the proper factor. Let the units be your guide and carefully use parentheses: $\left(\frac{7\pi}{6}\text{ radians}\right)\left(\frac{180^o}{\pi\text{ radians}}\right) = 210^o$



Note that the MODE setting does not matter because you are not using the ANGLE menu or any of the trig function keys. Next, we see how to have the TI-89 do the conversion.

Set MODE to Degree. Type 7π/6 and press [2nd] 5 (MATH) 2 [Angle] 2 (^{r}) [ENTER]. You must put 7π/6 in parentheses or the calculator assumes only the 6 is measured in radians.

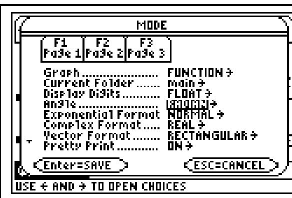


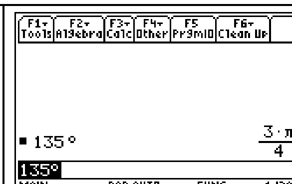
NOTE: When MODE is set to Degree, the TI-89 assumes all angles (inputs and outputs) are measured in degrees. The radian setting (^{r}) in the ANGLE menu overrides the MODE setting.

A.3 CONVERTING ANGLES FROM DEGREES TO RADIANS

We illustrate converting an angle from degree measure to radian measure using the angle in part *b* of Example 7 in the *Trigonometry Appendix*. As indicated in the last section, you can convert using the fact that π radians = 180 degrees or let the TI-89 do the conversion.

To have the TI-89 do the conversion, set MODE to Radian, type 135, and press [2nd] 5 (MATH) 2 [Angle] 1 (^{o}) [ENTER].





NOTE: When MODE is set to Radian, the TI-89 assumes all angles (inputs and outputs) are measured in radians. The degree choice (^{o}) in the ANGLE menu overrides the MODE setting.