Chapter 10 Analyzing Multivariable Change: Optimization

10.2 Multivariable Optimization

As you might expect, multivariable optimization techniques that you use with your TI-89 are very similar to those that were discussed in Chapter 5. The basic difference is that the algebra required to get the expression that comes from solving a system of equations with several unknowns reduced to one equation in one unknown is sometimes difficult. However, once your equation is of that form, all the optimization procedures are basically the same as those that were discussed previously.

10.2.1 **FINDING CRITICAL POINTS** Critical points for a multivariable function are points at which maxima, minima, or saddle points occur. To find critical point(s) of a smooth, continuous multivariable function, we first find the partial derivatives of the multivariable function with respect to each of the input variables and then set those partial derivatives equal to zero. This gives a *system of equations* to be solved. Most calculators solve linear systems of equations using an orderly array of numbers called a *matrix*. However, the TI-89 solve instruction can solve most systems of two equations if you use the syntax *solve(equation 1 and equation 2, {input variable, other input variable})*.

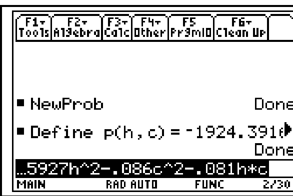
Consider the function $P(h, c)$ that gives the percentage of sugar converted to olestra when the ration of peanut oil to sugar is 10 to 1. This function is given at the beginning of Section 10.2 of *Calculus Concepts*:

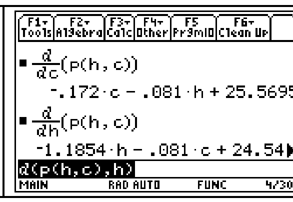
$$P(h, c) = {}^{-}1924.3916 + 24.5402h + 25.5695c - 0.5927h^2 - 0.0860c^2 - 0.0810hc$$

The system of equations derived from the partial derivatives of P are

$$P_c = 25.5695 - 0.172c - 0.081h = 0$$
$$P_h = 24.5402 - 1.1854h - 0.081c = 0$$

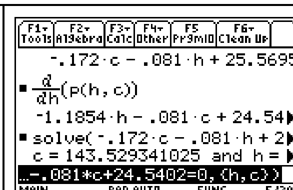
Either define* the function P (or enter it in the Y= list) because we later need to use it. First, let's verify the partial derivatives given above (or have the TI-89 find them).



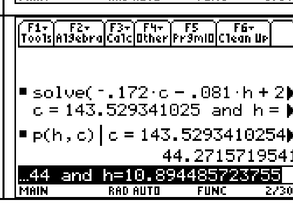


*Recall that *Define* is accessed with F4 [Other] 1 [Define]. Do not forget to put a times sign in the last term between h and c.

Now enter *solve* ($p_c = 0$ *and* $p_h = 0$, *{h, c}*). Access "{" with 2nd (and "}" with 2nd). It is easier (and probably more accurate) to copy and paste, editing with ←, the partial derivatives into the equations that equal 0 in the solve instruction.



We need to use these unrounded values to find the multivariable function output at the critical point. To do this, again use the copy and paste feature and the previous answer line to fill the space following "|" in the instruction *p(h, c)* | *h = a and c = b*.



NOTE: The solve instruction lists the solutions with *c* first and then *h*. You can paste this in behind "|" exactly as they are listed in the answer line. The order of the variables in the assignment statement does not matter as long as the numbers are identified with the variables.

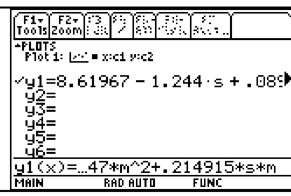
10.2.2 FINDING CRITICAL POINTS USING ALGEBRA AND THE SOLVER There will be times when the solve instruction fails to give an answer and returns the word "false". (This often happens when you are dealing with an exponential multivariable function.) This might mean that there is no solution or that the TI-89 cannot find the solution the way we are asking it to. Thus, you need to have available a method that works on any type of equations.

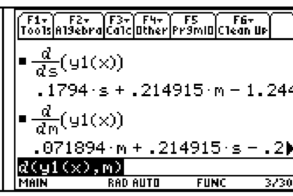
We illustrate using a combination of algebra and the solve instruction for the times when you need it. (In fact, we encounter such a case in Section 10.3.2.) Consider the function in Example 2 of Section 10.2.

> The total daily intake of organic matter required by a beef cow grazing the Northern Great Plains rangeland can be modeled by
>
> $$I(s, m) = 8.61967 - 1.244s + 0.0897s^2 - 0.20988m + 0.035947m^2 + 0.214915sm$$
>
> kg per day when the cow produces *m* kilograms of milk per day and *s* is a number between ⁻4 and 4 indicating the size of the cow.

This method is easier to use if you use the Y= list to hold the functions rather than defining on the home screen. Therefore, enter *I* (using the letters *s* and *m* as the function appears above) in y1 and refer to it on the home screen by the TI-89 name y1(x).

After entering the function in y1, find the partial derivative of *I* with respect to *s* and the partial derivative of *I* with respect to *m*.





Set each of the partial derivatives equal to 0 to obtain these equations:

$$I_s: \quad -1.244 + 0.1794s + 0.214915m = 0 \quad [1]$$

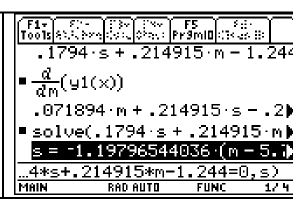
$$I_m: \quad -0.20988 + 0.071894m + 0.214915s = 0 \quad [2]$$

WARNING: Everything that you do with your calculator depends on the partial derivative formulas that you find using derivative rules. Double check this work and double check your typing of the partial derivatives before you use any of these solution methods. Also, if you define functions using single letters, be certain to clear the variables with each new problem.

Next, solve one of the equations for one of the variables, say equation 1 for *s*, to obtain

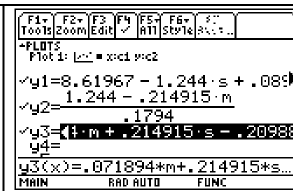
$$s = \frac{1.244 - 0.214915m}{0.1794}$$

The TI-89 will do the above solving for you with the instruction *solve(−1.244 + 0.1794s + 0.214915m = 0, s)*. You can copy the partial derivative into the instruction from the previous line. If you don't trust your algebra, let the TI-89 do it for you!

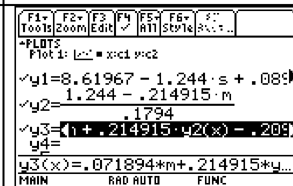


Recall that to *solve* for a quantity means that it must be by itself on one side of the equation and that the other side of the equation does not contain that letter. We use the quotient expression given above for *s*, but you could instead copy the calculator's result into the Y= list. Now, let your TI-89 work.

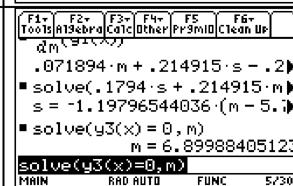
Enter the expression for *s* in y2. (Be sure to enclose the numerator in parentheses when the numerator is a quotient.) Copy on the home screen the left-hand side of the *other* equation (here, equation 2) and paste it in y3. Note that the expression you enter in y3 must be equal to zero.



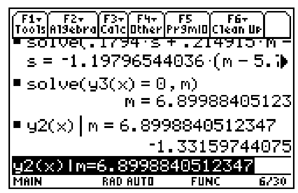
Now, put the cursor to the left of the letter *s* in the y3 location. Replace *every s* in y3 with y2(x). What you have just done is substitute the expression for *s* from equation 1 (which the TI-89 knows as y2) in equation 2!

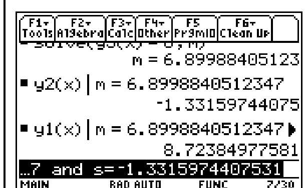


The expression in y3 is the left-hand side of an equation that equals 0 and contains only one variable, namely *m*. The next step is to solve the equation y3(x) = 0. Return to the home screen and use the solve instruction.



Return to the home screen and find *s* by evaluating y2 at the value of *m* you just found. The last thing to do is to find the output *I*(*s*, *m*) at the current values of *s* and *m*. (Use copy and paste with all numerical values.)





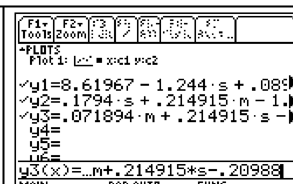
The critical point has coordinates $s \approx -1.33$, $m \approx 6.90$ kg of milk, and $I \approx 8.72$ kg of organic matter eaten each day.

10.2.3 CLASSIFYING CRITICAL POINTS USING THE DETERMINANT TEST

Once you find one or more critical points, the next step is to classify them as points at which a maximum, a minimum, or a saddle point occurs. The Determinant Test often will give the answer. Also, because this test uses derivatives, the calculator's derivative function can help.

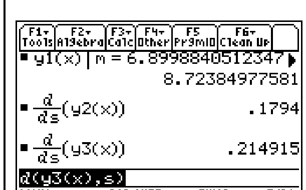
We illustrate with the critical point that was found for Example 2. To use the Determinant Test, we need to calculate the four second partial derivatives of *I* and evaluate them at the critical point values of *s* and *m*.

Enter the function *I* in y1, I_S in y2 and I_m in y3. These quantities are given on page C-98 of this *Guide*. (If you are continuing from the previous section, you need to edit y2 and y3.) Keep the unrounded values of the inputs (*m* and *s*) at the critical point on the home screen.



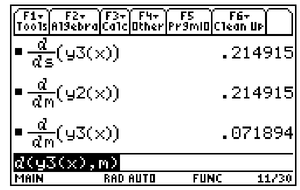
Take the derivative of y2 = I_S with respect to *s* and we have I_{SS}, the partial derivative of *I* with respect to *s* and then *s* again. Find $I_{SS} \approx 0.1794$ at the critical point.

Take the derivative of y3 = I_m with respect to *s* and we have I_{ms}, the partial derivative of *I* with respect to *m* and then *s*. Find $I_{ms} \approx 0.2149$ at the critical point.

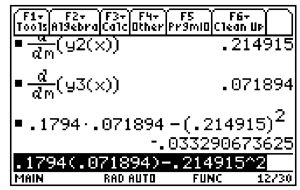


NOTE: If I_{ss}, I_{ms}, I_{sm}, and/or I_{mm} are not constant, evaluate them at the critical point values of m and s using the cut and paste feature of the TI-89. It is not advisable to round these values because a wrong result is then very possible when you are classifying the critical point.

Take the derivative of y2 = I_s with respect to m and we have I_{sm}, the partial derivative of I with respect to s and then m. Find $I_{sm} \approx 0.2149$ at the critical point.
Take the derivative of y3 = I_m with respect to m and we have I_{mm}, the partial derivative of I with respect to m and then m again. Find $I_{mm} \approx 0.0719$ at the critical point.



The second partials matrix is $\begin{bmatrix} I_{ss} & I_{sm} \\ I_{ms} & I_{mm} \end{bmatrix}$. Find the value of $D \approx (0.1794)(0.\ 71894) - (0.214915)^2 \approx -0.033$. Because $D < 0$, the Determinant Test tells us that $(s, m, I) \approx (-1.33, 6.90, 8.72)$ is a saddle point.



10.3 Optimization Under Constraints

Optimization techniques on your calculator when a constraint is involved are the same as the ones discussed in Sections 10.2.1 and 10.2.2 except that there is one additional equation in the system of equations to be solved.

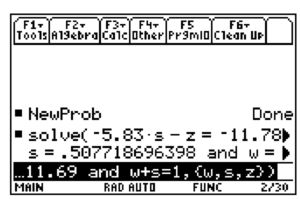
10.3.1 **FINDING OPTIMAL POINTS UNDER CONSTRAINED OPTIMIZATION** You can use any of the methods indicated in Sections 10.2.1 or 10.2.2 of this *Guide* to find critical points under constrained optimization. We choose to use the solve instruction to find the solution to the linear system of equations that results in the sausage cooking illustration that is at the beginning of Section 10.3 of *Calculus Concepts*. The system of equations, with partials found using the TI-89 or found by you using the partial derivative formulas, is

$$\begin{aligned} -5.83s - \lambda &= -11.78 \\ -5.83w - \lambda &= -11.69 \\ w + s &= 1 \end{aligned}$$

Even though you can type λ using Option 1: Greek in the CHAR menu (2nd +), we choose to type z for λ. Using l is not a good idea because it is too easily mistaken for the number 1.

Enter *solve(−5.83s – z = −11.78 and −5.83w – z = −11.69 and w + s = 1, {w, s, z})*. The solution $s \approx 0.508$, $w \approx 0.492$, and $z = \lambda = 8.82$ is found quickly.

Note that if the TI-89 seems to be taking a very long time to find the solution that you can press ON to halt the solve process and try again, typing *{w, s, z}* with the variables in a different order.



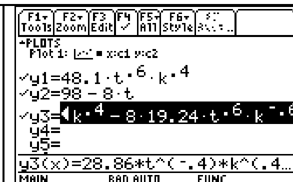
10.3.2 **CLASSIFYING OPTIMAL POINTS UNDER CONSTRAINED OPTIMIZATION** Classifying critical points when a constraint is involved is done by graphing the constraint on a contour graph or by examining close points. We illustrate the procedure used to examine close points for the function given in Example 1 of Section 10.3 of *Calculus Concepts* – the Cobb-

Douglas production function $f(L, K) = 48.1L^{0.6}K^{0.4}$ subject to the constraint $g(L, K) = 8L + K = 98$.

We first find the critical point(s). Unfortunately, if you attempt to solve the system of equations using the solve instruction (as we did in the previous illustration), the result "false" is returned no matter in what order the list of variables is entered. We therefore use the algebraic method discussed in Section 10.2.2. The system of equations is

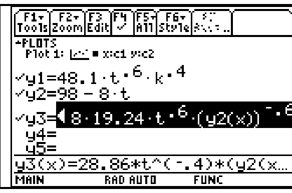
$$\left\{\begin{array}{l} \left.\begin{array}{l} 28.86L^{-0.4}K^{0.4} = 8\lambda \\ 19.24L^{0.6}K^{-0.6} = \lambda \end{array}\right\} \Rightarrow \quad 28.86L^{-0.4}K^{0.4} = 8(19.24L^{0.6}K \qquad [3] \\ 8L + K = 98 \qquad \Rightarrow \quad K = 98 - 8L \qquad [4] \end{array}\right.$$

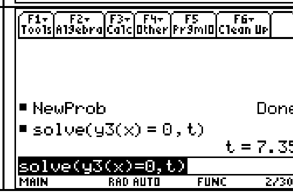
Clear the Y= list. Enter the function f in y1 and the expression for K (from equation 4) in y2. We choose to type t for the variable L. Rewrite the *other* equation (here, equation 3) so that it equals 0, and enter the non-zero side in y3.



Next, substitute* y2 into y3.

Use the solve instruction to solve the equation y3(x) = 0.

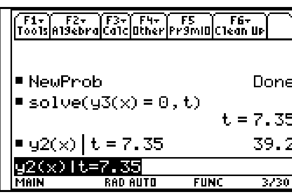


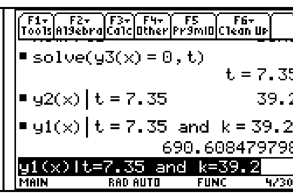


*Remember that K = y2. Put the cursor to the right of the first K in y3 and replace K by y2(x). Do the same for the other K in y3. The expression now in y3 is the left-hand side of an equation that equals 0 and contains only one variable, namely $t = L$. (We are not sure how many answers there are to this equation.)

Evaluate y2 = K at this value of t.

y1 gives the value of f at this point.



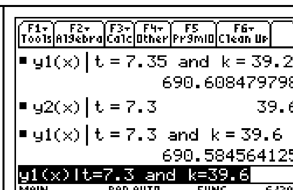


We now need to test close points to see if this output value of f is maximum or minimum. You need to remember that whatever close points you choose, they must be near the critical point and *they must be on the constraint g*.

WARNING: Do not round during this procedure. Rounding of intermediate calculations and/or inputs can give a false result when the close point is very near to the optimal point.

Choose a value of L that is less than $L = 7.35$, say 7.3. Find the value of K so that $8L + K = 98$. Remember that K = y2, so just evaluate y2 at 7.3. Then find f at $L = 7.3$, K = 39.6.

At this close point, the value of f is less than the value of f at the optimal point.



Choose another value of L, this time one that is more than $L = 7.35$, say 7.4. Find the value of K so that $8L + K = 98$ by evaluating y2 at 7.4. Find f at $L = 7.4$, $K = 38.8$.

At this close point, the value of f is less than the value of f at the critical point. Thus, (7.35, 39.2, 690) is a maximum point.