

- C** 25. (a) Compute the eigenvalues of

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}.$$

- (b) Apply four iterations of the power method with scaling to each matrix in part (a), starting with  $\mathbf{x}_0 = (-1, 2)$ .  
 (c) Compute the ratios  $\lambda_2/\lambda_1$  for  $A$  and  $B$ . For which do you expect faster convergence?

26. Use the proof of Theorem 10.3 to show that

$$A(A^k \mathbf{x}_0) \approx \lambda_1(A^k \mathbf{x}_0)$$

for large values of  $k$ . That is, show that the scale factors obtained in the power method approach the dominant eigenvalue.

- C** In Exercises 27 and 28, apply four iterations of the power method (with scaling) to approximate the dominant eigenvalue of the given matrix. After each iteration, scale the approximation by dividing by its length so that the resulting approximation will be a unit vector.

$$27. A = \begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix}$$

$$28. A = \begin{bmatrix} 7 & -4 & 2 \\ 16 & -9 & 6 \\ 8 & -4 & 5 \end{bmatrix}$$

## 10.4 APPLICATIONS OF NUMERICAL METHODS

### Applications of Gaussian Elimination with Pivoting

In Section 2.5 we used least squares regression analysis to find *linear* mathematical models that best fit a set of  $n$  points in the plane. This procedure can be extended to cover polynomial models of any degree as follows.

#### Regression Analysis for Polynomials

The least squares regression polynomial of degree  $m$  for the points  $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$  is given by

$$y = a_m x^m + a_{m-1} x^{m-1} + \dots + a_2 x^2 + a_1 x + a_0,$$

where the coefficients are determined by the following system of  $m + 1$  linear equations.

$$\begin{aligned} na_0 + (\sum x_i)a_1 + (\sum x_i^2)a_2 + \dots + (\sum x_i^m)a_m &= \sum y_i \\ (\sum x_i)a_0 + (\sum x_i^2)a_1 + (\sum x_i^3)a_2 + \dots + (\sum x_i^{m+1})a_m &= \sum x_i y_i \\ (\sum x_i^2)a_0 + (\sum x_i^3)a_1 + (\sum x_i^4)a_2 + \dots + (\sum x_i^{m+2})a_m &= \sum x_i^2 y_i \\ &\vdots \\ (\sum x_i^m)a_0 + (\sum x_i^{m+1})a_1 + (\sum x_i^{m+2})a_2 + \dots + (\sum x_i^{2m})a_m &= \sum x_i^m y_i \end{aligned}$$

Note that if  $m = 1$  this system of equations reduces to

$$\begin{aligned} na_0 + (\sum x_i)a_1 &= \sum y_i \\ (\sum x_i)a_0 + (\sum x_i^2)a_1 &= \sum x_i y_i, \end{aligned}$$

which has a solution of

$$a_1 = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2} \quad \text{and} \quad a_0 = \frac{\sum y_i}{n} - a_1 \frac{\sum x_i}{n}.$$

Exercise 16 asks you to show that this formula is equivalent to the matrix formula for linear regression that was presented in Section 2.5.

Example 1 illustrates the use of regression analysis to find a second-degree polynomial model.

### EXAMPLE 1 *Least Squares Regression Analysis*

The world population in billions for the years between 1950 and 1985, as given by the *Statistical Abstract of the United States*, is shown in the following table.

Year	1950	1955	1960	1965	1970	1975	1980	1985
Population	2.53	2.77	3.05	3.36	3.72	4.10	4.47	4.87

Find the second-degree least squares regression polynomial for these data and use the resulting model to predict the world population for 1990 and 1995.

**Solution** We begin by letting  $x = -4$  represent 1950,  $x = -3$  represent 1955, and so on. Thus the collection of points is given by  $\{(-4, 2.53), (-3, 2.77), (-2, 3.05), (-1, 3.36), (0, 3.72), (1, 4.10), (2, 4.47), (3, 4.87)\}$ , which yields

$$\begin{aligned} n = 8, \quad \sum_{i=1}^8 x_i = -4, \quad \sum_{i=1}^8 x_i^2 = 44, \quad \sum_{i=1}^8 x_i^3 = -64, \\ \sum_{i=1}^8 x_i^4 = 452, \quad \sum_{i=1}^8 y_i = 28.87, \quad \sum_{i=1}^8 x_i y_i = -0.24, \quad \sum_{i=1}^8 x_i^2 y_i = 146.78. \end{aligned}$$

Therefore the system of linear equations giving the coefficients of the quadratic model  $y = a_2 x^2 + a_1 x + a_0$  is

$$\begin{aligned} 8a_0 - 4a_1 + 44a_2 &= 28.87 \\ -4a_0 + 44a_1 - 64a_2 &= -0.24 \\ 44a_0 - 64a_1 + 452a_2 &= 146.78. \end{aligned}$$

Gaussian elimination with pivoting on the matrix

$$\begin{bmatrix} 8 & -4 & 44 & 28.87 \\ -4 & 44 & -64 & -0.24 \\ 44 & -64 & 452 & 146.78 \end{bmatrix}$$

produces

$$\begin{bmatrix} 1 & -1.4545 & 10.2727 & 3.3359 \\ 0 & 1 & -0.6000 & 0.3432 \\ 0 & 0 & 1 & 0.0130 \end{bmatrix}.$$

Thus by back substitution we find the solution to be

$$a_2 = 0.0130, \quad a_1 = 0.3510, \quad a_0 = 3.7126,$$

and the regression quadratic is

$$y = 0.0130x^2 + 0.3510x + 3.7126.$$

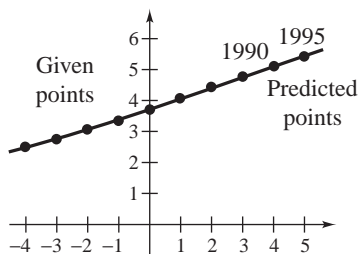
Figure 10.1 compares this model with the given points. To predict the world population for 1990, we let  $x = 4$ , obtaining

$$y = 0.0130(4^2) + 0.3510(4) + 3.7126 = 5.32 \text{ billion.}$$

Similarly, the prediction for 1995 ( $x = 5$ ) is

$$y = 0.0130(5^2) + 0.3510(5) + 3.7126 = 5.79 \text{ billion.}$$

Figure 10.1



### EXAMPLE 2 *Least Squares Regression Analysis*

Find the third-degree least squares regression polynomial

$$y = a_3x^3 + a_2x^2 + a_1x + a_0$$

for the points

$$\{(0, 0), (1, 2), (2, 3), (3, 2), (4, 1), (5, 2), (6, 4)\}.$$

**Solution** For this set of points the linear system

$$\begin{aligned} na_0 + (\Sigma x_i)a_1 + (\Sigma x_i^2)a_2 + (\Sigma x_i^3)a_3 &= \Sigma y_i \\ (\Sigma x_i)a_0 + (\Sigma x_i^2)a_1 + (\Sigma x_i^3)a_2 + (\Sigma x_i^4)a_3 &= \Sigma x_i y_i \\ (\Sigma x_i^2)a_0 + (\Sigma x_i^3)a_1 + (\Sigma x_i^4)a_2 + (\Sigma x_i^5)a_3 &= \Sigma x_i^2 y_i \\ (\Sigma x_i^3)a_0 + (\Sigma x_i^4)a_1 + (\Sigma x_i^5)a_2 + (\Sigma x_i^6)a_3 &= \Sigma x_i^3 y_i \end{aligned}$$

becomes

$$\begin{aligned} 7a_0 + 21a_1 + 91a_2 + 441a_3 &= 14 \\ 21a_0 + 91a_1 + 441a_2 + 2,275a_3 &= 52 \\ 91a_0 + 441a_1 + 2,275a_2 + 12,201a_3 &= 242 \\ 441a_0 + 2,275a_1 + 12,201a_2 + 67,171a_3 &= 1,258. \end{aligned}$$

Using Gaussian elimination with pivoting on the matrix

$$\begin{bmatrix} 7 & 21 & 91 & 441 & 14 \\ 21 & 91 & 441 & 2,275 & 52 \\ 91 & 441 & 2,275 & 12,201 & 242 \\ 441 & 2,275 & 12,201 & 67,171 & 1,258 \end{bmatrix}$$

produces

$$\begin{bmatrix} 1.0000 & 5.1587 & 27.6667 & 152.3150 & 2.8526 \\ 0.0000 & 1.0000 & 8.5312 & 58.3482 & 0.6183 \\ 0.0000 & 0.0000 & 1.0000 & 9.7714 & 0.1286 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.1667 \end{bmatrix},$$

which implies that

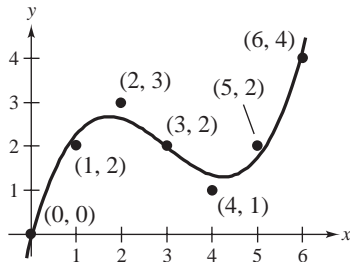
$$a_3 = 0.1667, \quad a_2 = -1.5000, \quad a_1 = 3.6905, \quad a_0 = -0.0714.$$

Therefore the cubic model is

$$y = 0.1667x^3 - 1.5000x^2 + 3.6905x - 0.0714.$$

Figure 10.2 compares this model with the given points.

Figure 10.2



## Applications of the Gauss-Seidel Method

### EXAMPLE 3 An Application to Probability

Figure 10.3 is a diagram of a maze used in a laboratory experiment. The experiment is begun by placing a mouse at one of the ten interior intersections of the maze. Once the mouse emerges in the outer corridor, it cannot return to the maze. When the mouse is at an interior intersection, its choice of paths is assumed to be random. What is the probability that the mouse will emerge in the “food corridor” when it begins at the  $i$ th intersection?

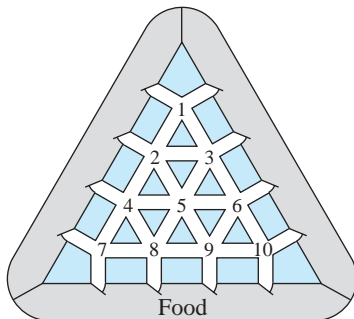
**Solution**

We let the probability of winning (getting food) by starting at the  $i$ th intersection be represented by  $p_i$ . Then we form a linear equation involving  $p_i$  and the probabilities associated with the intersections bordering the  $i$ th intersection. For instance, at the first intersection the mouse has a probability of  $\frac{1}{4}$  of choosing the upper right path and losing, a probability of  $\frac{1}{4}$  of choosing the upper left path and losing, a probability of  $\frac{1}{4}$  of choosing the lower right path (at which point it has a probability of  $p_3$  of winning), and a probability of  $\frac{1}{4}$  of choosing the lower left path (at which point it has a probability of  $p_2$  of winning). Thus we have

$$p_1 = \frac{1}{4}(0) + \frac{1}{4}(0) + \frac{1}{4}p_2 + \frac{1}{4}p_3.$$

$\underbrace{\hspace{1.5em}}$   
Upper  
right
 $\underbrace{\hspace{1.5em}}$   
Upper  
left
 $\underbrace{\hspace{1.5em}}$   
Lower  
left
 $\underbrace{\hspace{1.5em}}$   
Lower  
right

Figure 10.3



Using similar reasoning, we find the other nine probabilities to be represented by the following equations.

$$\begin{aligned}
 p_2 &= \frac{1}{5}(0) + \frac{1}{5}p_1 + \frac{1}{5}p_3 + \frac{1}{5}p_4 + \frac{1}{5}p_5 \\
 p_3 &= \frac{1}{5}(0) + \frac{1}{5}p_1 + \frac{1}{5}p_2 + \frac{1}{5}p_5 + \frac{1}{5}p_6 \\
 p_4 &= \frac{1}{5}(0) + \frac{1}{5}p_2 + \frac{1}{5}p_5 + \frac{1}{5}p_7 + \frac{1}{5}p_8 \\
 p_5 &= \frac{1}{6}p_2 + \frac{1}{6}p_3 + \frac{1}{6}p_4 + \frac{1}{6}p_6 + \frac{1}{6}p_8 + \frac{1}{6}p_9 \\
 p_6 &= \frac{1}{5}(0) + \frac{1}{5}p_3 + \frac{1}{5}p_5 + \frac{1}{5}p_9 + \frac{1}{5}p_{10} \\
 p_7 &= \frac{1}{4}(0) + \frac{1}{4}(1) + \frac{1}{4}p_4 + \frac{1}{4}p_8 \\
 p_8 &= \frac{1}{5}(1) + \frac{1}{5}p_4 + \frac{1}{5}p_5 + \frac{1}{5}p_7 + \frac{1}{5}p_9 \\
 p_9 &= \frac{1}{5}(1) + \frac{1}{5}p_5 + \frac{1}{5}p_6 + \frac{1}{5}p_8 + \frac{1}{5}p_{10} \\
 p_{10} &= \frac{1}{4}(0) + \frac{1}{4}(1) + \frac{1}{4}p_6 + \frac{1}{4}p_9
 \end{aligned}$$

Rewriting these equations in standard form produces the following system of ten linear equations in ten variables.

$$\begin{aligned}
 4p_1 - p_2 - p_3 &= 0 \\
 -p_1 + 5p_2 - p_3 - p_4 - p_5 &= 0 \\
 -p_1 - p_2 + 5p_3 - p_5 - p_6 &= 0 \\
 -p_2 + 5p_4 - p_5 - p_7 - p_8 &= 0 \\
 -p_2 - p_3 - p_4 + 6p_5 - p_6 - p_8 - p_9 &= 0 \\
 -p_3 - p_5 + 5p_6 - p_9 - p_{10} &= 0 \\
 -p_4 + 4p_7 - p_8 &= 1 \\
 -p_4 - p_5 - p_7 + 5p_8 - p_9 &= 1 \\
 -p_5 - p_6 - p_8 + 5p_9 - p_{10} &= 1 \\
 -p_6 - p_9 + 4p_{10} &= 1
 \end{aligned}$$

The augmented matrix for this system is

$$\left[ \begin{array}{cccccccccc|c}
 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -1 & 5 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -1 & -1 & 5 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
 0 & -1 & 0 & 5 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\
 0 & -1 & -1 & -1 & 6 & -1 & 0 & -1 & -1 & 0 & 0 \\
 0 & 0 & -1 & 0 & -1 & 5 & 0 & 0 & -1 & -1 & 0 \\
 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 & 0 & 1 \\
 0 & 0 & 0 & -1 & -1 & 0 & -1 & 5 & -1 & 0 & 1 \\
 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 5 & -1 & 1 \\
 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 1
 \end{array} \right].$$

Using the Gauss-Seidel method with an initial approximation of  $p_1 = p_2 = \dots = p_{10} = 0$  produces (after 18 iterations) an approximation of

$$\begin{array}{ll} p_1 = 0.090, & p_2 = 0.180 \\ p_3 = 0.180, & p_4 = 0.298 \\ p_5 = 0.333, & p_6 = 0.298 \\ p_7 = 0.455, & p_8 = 0.522 \\ p_9 = 0.522, & p_{10} = 0.455. \end{array}$$

The structure of the probability problem described in Example 3 is related to a technique called **finite element analysis**, which is used in many engineering problems.

Note that the matrix developed in Example 3 has mostly zero entries. We call such matrices **sparse**. For solving systems of equations with sparse coefficient matrices, the Jacobi and Gauss-Seidel methods are much more efficient than Gaussian elimination.

## Applications of the Power Method

Section 7.4 introduced the idea of an *age transition matrix* as a model for population growth. Recall that this model was developed by grouping the population into  $n$  age classes of equal duration. Thus for a maximum life span of  $L$  years, the age classes are given by the following intervals.

$$\underbrace{\left[0, \frac{L}{n}\right)}_{\text{First age class}}, \quad \underbrace{\left[\frac{L}{n}, \frac{2L}{n}\right)}_{\text{Second age class}}, \quad \dots, \quad \underbrace{\left[\frac{(n-1)L}{n}, L\right)}_{\text{nth age class}}$$

The number of population members in each age class is then represented by the age distribution vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \begin{array}{l} \text{Number in first age class} \\ \text{Number in second age class} \\ \text{Number in } n\text{th age class} \end{array}$$

Over a period of  $L/n$  years, the *probability* that a member of the  $i$ th age class will survive to become a member of the  $(i + 1)$ th age class is given by  $p_i$ , where  $0 \leq p_i \leq 1$ ,  $i = 1, 2, \dots, n - 1$ . The *average number* of offspring produced by a member of the  $i$ th age class is given by  $b_i$ , where  $0 \leq b_i$ ,  $i = 1, 2, \dots, n$ . These numbers can be written in matrix form:

$$A = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ p_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1} & 0 \end{bmatrix}$$



To apply the power method with scaling to find an eigenvector for this matrix, we use an initial approximation of  $\mathbf{x}_0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ . The following is an approximation for an eigenvector of  $A$ , with the percentage of each age in the total population.

<i>Eigenvector</i>	<i>Age Class</i>	<i>Percentage in Age Class</i>
$\mathbf{x} = \begin{bmatrix} 1.000 \\ 0.925 \\ 0.864 \\ 0.806 \\ 0.749 \\ 0.686 \\ 0.605 \\ 0.492 \\ 0.314 \\ 0.106 \end{bmatrix}$	$0 \leq \text{age} < 10$	15.27
	$10 \leq \text{age} < 20$	14.13
	$20 \leq \text{age} < 30$	13.20
	$30 \leq \text{age} < 40$	12.31
	$40 \leq \text{age} < 50$	11.44
	$50 \leq \text{age} < 60$	10.48
	$60 \leq \text{age} < 70$	9.24
	$70 \leq \text{age} < 80$	7.51
	$80 \leq \text{age} < 90$	4.80
	$90 \leq \text{age} < 100$	1.62

---

The eigenvalue corresponding to the eigenvector  $\mathbf{x}$  in Example 4 is  $\lambda \approx 1.065$ . That is,

$$A\mathbf{x} = A \begin{bmatrix} 1.000 \\ 0.925 \\ 0.864 \\ 0.806 \\ 0.749 \\ 0.686 \\ 0.605 \\ 0.492 \\ 0.314 \\ 0.106 \end{bmatrix} \approx \begin{bmatrix} 1.065 \\ 0.985 \\ 0.921 \\ 0.859 \\ 0.798 \\ 0.731 \\ 0.645 \\ 0.524 \\ 0.334 \\ 0.113 \end{bmatrix} \approx 1.065 \begin{bmatrix} 1.000 \\ 0.925 \\ 0.864 \\ 0.806 \\ 0.749 \\ 0.686 \\ 0.605 \\ 0.492 \\ 0.314 \\ 0.106 \end{bmatrix}.$$

This means that the population in Example 4 increases by 6.5% every ten years.

REMARK: Should you try duplicating the results of Example 4, you would notice that the convergence of the power method for this problem is very slow. The reason is that the dominant eigenvalue of  $\lambda \approx 1.065$  is only slightly larger in absolute value than the next largest eigenvalue.

## SECTION 10.4 EXERCISES

## Applications of Gaussian Elimination with Pivoting

**C** In Exercises 1–4, find the second-degree least squares regression polynomial for the given data. Then graphically compare the model to the given points.

- $(-2, 1), (-1, 0), (0, 0), (1, 1), (3, 2)$
- $(0, 4), (1, 2), (2, -1), (3, 0), (4, 1), (5, 4)$
- $(-2, 1), (-1, 2), (0, 6), (1, 3), (2, 0), (3, -1)$
- $(1, 1), (2, 1), (3, 0), (4, -1), (5, -4)$

**C** In Exercises 5–8, find the third-degree least squares regression polynomial for the given data. Then graphically compare the model to the given points.

- $(0, 0), (1, 2), (2, 4), (3, 1), (4, 0), (5, 1)$
- $(1, 1), (2, 4), (3, 4), (5, 1), (6, 2)$
- $(-3, 4), (-1, 1), (0, 0), (1, 2), (2, 5)$
- $(-7, 2), (-3, 0), (1, -1), (2, 3), (4, 6)$

**C** 9. Find the second-degree least squares regression polynomial for the points

$$\left(-\frac{\pi}{2}, 0\right), \left(-\frac{\pi}{3}, \frac{1}{2}\right), (0, 1), \left(\frac{\pi}{3}, \frac{1}{2}\right), \left(\frac{\pi}{2}, 0\right).$$

Then use the results to approximate  $\cos(\pi/4)$ . Compare the approximation with the exact value.

**C** 10. Find the third-degree least squares regression polynomial for the points

$$\left(-\frac{\pi}{4}, -1\right), \left(-\frac{\pi}{3}, -\sqrt{3}\right), (0, 0), \left(\frac{\pi}{3}, \sqrt{3}\right), \left(\frac{\pi}{4}, 1\right).$$

Then use the result to approximate  $\tan(\pi/6)$ . Compare the approximation with the exact value.

**C** 11. The number of minutes a scuba diver can stay at a particular depth without acquiring decompression sickness, as given by the United States Navy's Standard Air Decompression Tables, is shown in the following table.

Depth (in feet)	35	40	50	60	70
Time (in minutes)	310	200	100	60	50
Depth (in feet)	80	90	100	110	
Time (in minutes)	40	30	25	20	

- Find the least squares regression line for these data.
- Find the second-degree least squares regression polynomial for these data.
- Sketch the graphs of the models found in parts (a) and (b).
- Use the models found in parts (a) and (b) to approximate the maximum number of minutes a diver should stay at a depth of 120 feet. (The value given in the Navy's tables is 15 minutes.)

**C** 12. The life expectancy for *additional* years of life for females in the United States is given in the following table.

Current Age	10	20	30	40
Life Expectancy	69.7	59.9	50.2	40.6
Current Age	50	60	70	80
Life Expectancy	31.3	22.8	15.3	9.0

Find the second-degree least squares regression polynomial for these data.

**C** 13. Total federal payments (in billions of dollars) for Medicare from 1970 to 1984 are given in the following table.

Year	1970	1972	1974	1976
Medicare	7.5	9.1	13.1	19.3
Year	1978	1980	1982	1984
Medicare	25.9	36.8	52.4	64.8

Find the second-degree least squares regression polynomial for these data. Then use the result to predict the Medicare payments for 1986 and 1988.

**C** 14. Rework Exercise 13 using the data in the following table, in which Medicare payments have been adjusted for inflation.

Year	1970	1972	1974	1976
Medicare	7.5	8.4	10.3	13.2
Year	1978	1980	1982	1984
Medicare	15.4	17.3	21.1	24.2

- C** 15. Find the least squares regression line for the population data given in Example 1. Then use the model to predict the world population in 1990 and 1995, and compare the results with the predictions obtained in Example 1.
16. Show that the formula for the least squares regression line presented in Section 2.5 is equivalent to the formula presented in this section. That is, if

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix},$$

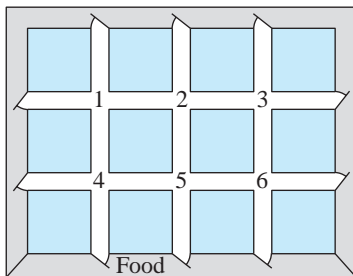
then the matrix equation  $A = (X^T X)^{-1} X^T Y$  is equivalent to

$$a_1 = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2} \quad \text{and} \quad a_0 = \frac{\sum y_i}{n} - a_1 \frac{\sum x_i}{n}.$$

### Applications of the Gauss-Seidel Method

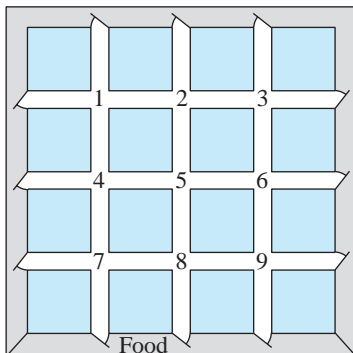
- C** 17. Suppose that the experiment in Example 3 is performed with the maze shown in Figure 10.4. Find the probability that the mouse will emerge in the food corridor when it begins in the  $i$ th intersection.

Figure 10.4



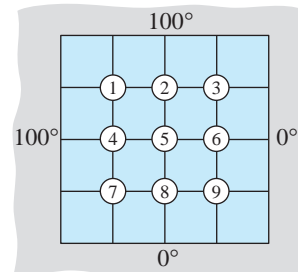
- C** 18. Suppose that the experiment in Example 3 is performed with the maze shown in Figure 10.5. Find the probability that the mouse will emerge in the food corridor when it begins in the  $i$ th intersection.

Figure 10.5



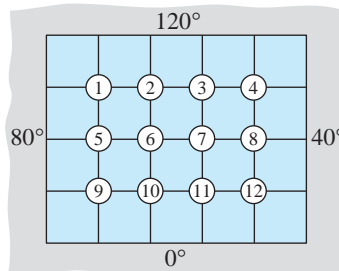
- C** 19. A square metal plate has a constant temperature on each of its four boundaries, as shown in Figure 10.6. Use a  $4 \times 4$  grid to approximate the temperature distribution in the interior of the plate. Assume that the temperature at each interior point is the average of the temperatures at the four closest neighboring points.

Figure 10.6



- C** 20. A rectangular metal plate has a constant temperature on each of its four boundaries, as shown in Figure 10.7. Use a  $4 \times 5$  grid to approximate the temperature distribution in the interior of the plate. Assume that the temperature at each interior point is the average of the temperatures at the four closest neighboring points.

Figure 10.7



### Applications of the Power Method

In Exercises 21–24, the given matrix represents the age transition matrix for a population. Use the power method with scaling to find a stable age distribution.

21.  $A = \begin{bmatrix} 1 & 4 \\ \frac{1}{2} & 0 \end{bmatrix}$

22.  $A = \begin{bmatrix} 1 & 2 \\ \frac{1}{4} & 0 \end{bmatrix}$

**C** 23.  $A = \begin{bmatrix} 0 & 1 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$

**C** 24.  $A = \begin{bmatrix} 1 & 2 & 2 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$

- C** 25. In Example 1 of Section 7.4, a laboratory population of rabbits is described. The age transition matrix for the population is

$$A = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

Find a stable age distribution for this population.

- C** 26. A population has the following characteristics.
- A total of 75% of the population survives its first year. Of that 75%, 25% survives its second year. The maximum life span is three years.
  - The average number of offspring for each member of the population is 2 the first year, 4 the second year, and 2 the third year.

Find a stable age distribution for this population. (See Exercise 9, Section 7.4.)

27. Apply the power method to the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

discussed in Chapter 7 (Fibonacci sequence). Use the power method to approximate the dominant eigenvalue of  $A$ . (The dominant eigenvalue is  $\lambda = (1 + \sqrt{5})/2$ .)

- C** 28. In Example 2 of Section 2.5, the stochastic matrix

$$P = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix}$$

represents the transition probabilities for a consumer preference model. Use the power method to approximate a dominant eigenvector for this matrix. How does the approximation relate to the steady-state matrix described in the discussion following Example 3 in Section 2.5?

- C** 29. In Exercise 9 of Section 2.5, a population of 10,000 is divided into nonsmokers, moderate smokers, and heavy smokers. Use the power method to approximate a dominant eigenvector for this matrix.