71. Use mathematical induction to prove DeMoivre's Theorem.

72. Prove that if z is a zero of a polynomial equation with real coefficients, then the conjugate of z is also a zero.

73. Show that if $z_1 + z_2$ and $z_1\overline{z_2}$ are both nonzero real numbers, then z_1 and z_2 are both real numbers.

74. Prove that if z and w are complex numbers, then

$$|z + w| \leq |z| + |w|.$$

75. Prove that for all vectors $\mathbf{u}$ and $\mathbf{v}$ in a complex inner product space,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \tfrac{1}{4}\big[\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 + i\|\mathbf{u} + i\mathbf{v}\|^2 - i\|\mathbf{u} - i\mathbf{v}\|^2\big].$$

CHAPTER 8 PROJECTS

1 Population Growth and Dynamical Systems - II

In the projects for Chapter 7, you were asked to model the population of two species using a system of differential equations of the form

$$y_1'(t) = ay_1(t) + by_2(t)$$
$$y_2'(t) = cy_1(t) + dy_2(t).$$

The constants a, b, c, and d depend on the particular species being studied. In Chapter 7, we looked at an example of a predator–prey relationship, in which $a = 0.5$, $b = 0.6$, $c = -0.4$, and $d = 3.0$. Suppose we now consider a slightly different model.

$$y_1'(t) = 0.6y_1(t) + 0.8y_2(t), \quad y_1(0) = 36$$
$$y_2'(t) = -0.8y_1(t) + 0.6y_2(t), \quad y_2(0) = 121$$

1. Use the diagonalization technique to find the general solutions $y_1(t)$ and $y_2(t)$ at any time $t > 0$. Although the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix}$$

are complex, the same principles apply, and you can obtain complex exponential solutions.

2. Convert your complex solutions to real solutions by observing that if $\lambda = a + bi$ is a (complex) eigenvalue of A with (complex) eigenvector **v,** then the real and imaginary parts of $e^{\lambda t}\mathbf{v}$ form a linearly independent pair of (real) solutions. You will need to use the formula $e^{i\theta} = \cos\theta + i\sin\theta$.
3. Use the initial conditions to find the explicit form of the (real) solutions to the original equations.
4. If you have access to a computer or graphing calculator, graph the solutions obtained in part (3) over the domain $0 \leq t \leq 3$. At what moment are the two populations equal?
5. Interpret the solution in terms of the long-term population trend for the two species. Does one species ultimately disappear? Why or why not? Contrast this solution to that obtained for the model in Chapter 7.
6. If you have access to a computer or graphing calculator that can numerically solve differential equations, use it to graph the solutions to the original system of equations. Does this numerical approximation appear to be accurate?

CHAPTER 8 ❑ MATLAB EXERCISES

1. MATLAB handles complex numbers and matrices in much the same way as real ones. The imaginary unit $i = \sqrt{-1}$ is a built-in constant. For instance, the complex number $2 - 3i$ would be represented as $2 - 3 * i$ in MATLAB. You can verify the result of Example 5, Section 8.2, by entering the matrix A,

$A = [2 - i \quad -5 + 2 * i \;\; ; \;\; 3 - i \quad -6 + 2 * i]$

and then typing **inv(A).**

Use MATLAB to perform the following matrix operations, given

$$A = \begin{bmatrix} 1 & 2 - i \\ 2 + i & i \end{bmatrix}, \; B = \begin{bmatrix} 3i & 4 \\ -4 & -i \end{bmatrix}, \text{ and } \; C = \begin{bmatrix} i & -i & 0 \\ 2 & 0 & 2 + 3i \end{bmatrix}.$$

(a) AB (b) $3iC$ (c) A^{-1}

(d) C^TC (e) $|A + B|$ (f) $iAB^2 + (1 - i)CC^T$

2. Use MATLAB to solve the system of linear equations $A\mathbf{x} = \mathbf{b}$.

(a) $A = \begin{bmatrix} i & 2 - i \\ 3 - 2i & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 + i \\ 6 - 4i \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & i \\ -2 & i + 1 & -i \\ 1 - i & 0 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} i \\ 0 \\ 2 - i \end{bmatrix}$

3. For a complex matrix A, the MATLAB command **A'** produces the complex conjugate transpose A^* of A. Use this command to determine which of the following matrices are Hermitian and which are normal.

(a) $\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 - 4i & 2 \\ 1 + i & 4 - i \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & i \\ 0 & i & 2 \end{bmatrix}$

4. The MATLAB command $[\mathbf{P,D}] = \mathbf{eig}(\mathbf{A})$ will produce a diagonal matrix D containing the eigenvalues of the complex matrix A, and a matrix P containing the corresponding eigenvectors. For instance, if

$$A = \begin{bmatrix} 3 & 2 - i & -3i \\ 2 + i & 0 & 1 - i \\ 3i & 1 + i & 0 \end{bmatrix}$$

is the matrix from Example 7, Section 8.5, then the command $[\mathbf{P,D}] = \mathbf{eig}(\mathbf{A})$ yields

$$P = \begin{bmatrix} 0.7792 + 0.0000i & -0.4472 + 0.2236i & -0.2870 - 0.2460i \\ 0.3438 + 0.2063i & 0.1118 - 0.3354i & -0.2050 + 0.8199i \\ 0.0229 + 0.4813i & 0.1118 + 0.7826i & 0.2870 + 0.2460i \end{bmatrix}$$

and

$$D = \begin{bmatrix} 6.0000 + 0.0000i & 0 & 0 \\ 0 & -2.0000 + 0.0000i & 0 \\ 0 & 0 & -1.0000 - 0.0000i \end{bmatrix},$$

which is equivalent to the solution given in the text. Use this procedure to diagonalize the following matrices.

(a) $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & i \\ 0 & i & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1+i & 1-i \\ 1-i & 0 & i \\ 1+i & -i & 0 \end{bmatrix}$